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I consider infinitesimal translations  $x'^\alpha = x^\alpha + \delta x^\alpha$  and demand that Noether's approach gives a symmetric energy-momentum tensor as it is required for gravitational sources. This argument determines the transformations of non-abelian gauge fields under infinitesimal translations to differ from the usually assumed invariance by the gauge transformation,  $A'^a_\gamma(x') - A^a_\gamma(x) = \partial_\gamma[\delta x_\beta A^{a\beta}(x)] + C^a_{bc} \delta x_\beta A^{c\beta}(x) A^b_\gamma(x)$  where the  $C^a_{bc}$  are the structure constants of the gauge group.

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In a previous paper [1] I have determined the transformations of the electromagnetic potentials under translations from the requirement that the energy-momentum tensor as it comes out of Noether's theorem [2,3] ought to be symmetric. Such a result is desirable, because the energy-momentum tensor enters as source of the gravitational field and the symmetry transformations of general covariance yield a symmetric tensor, see for instance [4,5]. A more detailed motivation is given in my first paper. Here I extend the argument to non-abelian gauge theories.

Following the notation of Weinberg's book [6], the Lagrange density is

$$\mathcal{L} = -\frac{1}{4} F^a_{\alpha\beta} F^{a\alpha\beta} \quad (1)$$

with

$$F^a_{\alpha\beta} = \partial_\alpha A^a_\beta - \partial_\beta A^a_\alpha + C^a_{bc} A^b_\alpha A^c_\beta \quad (2)$$

where the  $C^a_{bc}$  are the structure constants of the gauge group. The Lagrangian (1) is invariant under the gauge transformations of the fields:

$$A^a_\alpha \mapsto A^a_\alpha + \partial_\alpha \epsilon^a(x) + C^a_{bc} \epsilon^c(x) A^b_\alpha. \quad (3)$$

As in [1] the observation is that the conventionally assumed invariance of the gauge fields under translations

$$x'^\alpha = x^\alpha + \delta x^\alpha \quad (4)$$

can be enlarged by gauge transformations and the general form reads

$$A'^a_\gamma(x') = A^a_\gamma(x) + \partial_\gamma \epsilon^a(x) + C^a_{bc} \epsilon^c(x) A^b_\gamma. \quad (5)$$

Repeating the arguments of Noether's theorem in the version of [3] and requesting a symmetric energy-momentum

tensor determines the gauge transformation uniquely and leads to the transformation law stated in the abstract

$$A'^a_\gamma(x') = A^a_\gamma(x) + \partial_\gamma[\delta x_\beta A^{a\beta}(x)] + C^a_{bc} \delta x_\beta A^{c\beta}(x) A^b_\gamma(x). \quad (6)$$

The remainder of this letter is devoted to the derivation of this equation and my treatment follows closely [1] where also a few additional steps can be found.

First, let us consider general fields  $\psi_k$  and recall the derivation of the relativistic Euler Lagrange equations from the action principle. The action is a four dimensional integral over a scalar Lagrangian density

$$\mathcal{A} = \int d^4x \mathcal{L}(\psi_k, \partial_\alpha \psi_k). \quad (7)$$

Variations of the fields are defined as functions

$$\delta\psi_k(x) = \psi'_k(x) - \psi_k(x) \quad (8)$$

which are non-zero for some localized space-time region. The action is required to vanish under such variations

$$0 = \delta\mathcal{A} = \sum_k \int d^4x \left[ (\delta\psi_k) \frac{\partial\mathcal{L}}{\partial\psi_k} + (\delta\partial_\alpha\psi_k) \frac{\partial\mathcal{L}}{\partial(\partial_\alpha\psi_k)} \right]. \quad (9)$$

Integration by parts allows to factor  $\delta\psi_k$  out and, because all the  $\delta\psi_k$  are independent, we arrive at the Euler-Lagrange equations

$$\frac{\partial\mathcal{L}}{\partial\psi_k} - \partial_\alpha \frac{\partial\mathcal{L}}{\partial(\partial_\alpha\psi_k)} = 0. \quad (10)$$

Together with the anti-symmetry of  $F^a_{\alpha\beta}$  in the Lorentz indices, the Euler-Lagrange equations imply the relation

$$\partial_\gamma \frac{\partial\mathcal{L}}{\partial(\partial_\alpha A^a_\gamma)} = C^b_{ac} A^c_\beta \frac{\partial\mathcal{L}}{\partial(\partial_\beta A^b_\alpha)}. \quad (11)$$

Noether's theorem applies to transformations of the coordinates for which the transformations of the field functions are also known and we introduce, in addition to (8), a second type of variations which combines space-time and their corresponding field variations

$$\bar{\delta}\psi_k(x) = \psi'_k(x') - \psi_k(x). \quad (12)$$

Using

$$\psi'_k(x') = \psi'_k(x) + \delta x^\alpha \partial_\alpha \psi_k(x)$$

we find a relation between the variations (12) and (8)

$$\bar{\delta}\psi_k(x) = \delta\psi_k(x) + \delta x^\alpha \partial_\alpha \psi_k(x). \quad (13)$$

For a scalar field  $\psi$  symmetry under translations means

$$\bar{\delta}\psi(x) = \psi'(x') - \psi(x) = 0. \quad (14)$$

But for the gauge fields we allow (5)

$$\begin{aligned} \bar{\delta}A^a{}_\gamma(x) &= A'^a{}_\gamma(x') - A^a{}_\gamma(x) \\ &= \partial_\gamma \epsilon^a(x) + C^a{}_{bc} \epsilon^c(x) A^b{}_\gamma(x) \end{aligned} \quad (15)$$

and equation (13) becomes

$$\delta A^a{}_\gamma = \partial_\gamma \epsilon^a(x) + C^a{}_{bc} \epsilon^c(x) A^b{}_\gamma - \delta x^\alpha \partial_\alpha A^a{}_\gamma(x). \quad (16)$$

As the Lagrange density is a scalar, we get for its combined variation (12)

$$0 = \bar{\delta}\mathcal{L} = \mathcal{L}'(x') - \mathcal{L}(x) = \delta\mathcal{L} + \delta x^\alpha \partial_\alpha \mathcal{L} \quad (17)$$

where besides (14) we used the relation (13). Our aim is to factor an over-all variation  $\delta x^\alpha$  out. For  $\delta\mathcal{L}$  we proceed as in equation (9), where the  $\psi_k$  fields are now replaced by the gauge fields  $A^a{}_\gamma$

$$\delta\mathcal{L} = (\delta A^a{}_\gamma) \frac{\partial\mathcal{L}}{\partial A^a{}_\gamma} + (\delta\partial_\alpha A^a{}_\gamma) \frac{\partial\mathcal{L}}{\partial(\partial_\alpha A^a{}_\gamma)}.$$

Using the Euler-Lagrange equation (10) to eliminate  $\partial\mathcal{L}/\partial A^a{}_\gamma$ , we get

$$\delta\mathcal{L} = \partial_\alpha \left[ (\delta A^a{}_\gamma) \frac{\partial\mathcal{L}}{\partial(\partial_\alpha A^a{}_\gamma)} \right].$$

Let us collect all terms which contribute to  $\bar{\delta}\mathcal{L}$  in equation (17). We find (note that  $\partial_\beta \delta x^\alpha = 0$  holds for all combinations of indices  $\alpha, \beta$ )

$$\begin{aligned} 0 = \bar{\delta}\mathcal{L} &= \partial_\alpha \left[ (\delta A^a{}_\gamma) \frac{\partial\mathcal{L}}{\partial(\partial_\alpha A^a{}_\gamma)} + \delta x^\alpha \mathcal{L} \right] = \\ &\partial_\alpha \left[ (\partial_\gamma \epsilon^a(x) + C^a{}_{bc} \epsilon^c(x) A^b{}_\gamma) \frac{\partial\mathcal{L}}{\partial(\partial_\alpha A^a{}_\gamma)} \right] \\ &+ \delta x_\beta \partial_\alpha \left[ -(\partial^\beta A^a{}_\gamma) \frac{\partial\mathcal{L}}{\partial(\partial_\alpha A^a{}_\gamma)} + g^{\alpha\beta} \mathcal{L} \right] \end{aligned}$$

where equation (16) was used. To be able to factor  $\delta x_\beta$  also out of the first bracket on the right-hand side, one has to request

$$\epsilon^a(x) = \delta x_\beta B^{a\beta}(x) \quad (18)$$

where  $B^{a\beta}(x)$  is a not yet determined gauge field. With this we get

$$\begin{aligned} 0 &= \delta x_\beta \partial_\alpha \left[ (\partial^\beta A^a{}_\gamma) \frac{\partial\mathcal{L}}{\partial(\partial_\alpha A^a{}_\gamma)} \right. \\ &\quad \left. - (\partial_\gamma B^{a\beta} + C^a{}_{bc} B^{c\beta} A^b{}_\gamma) \frac{\partial\mathcal{L}}{\partial(\partial_\alpha A^a{}_\gamma)} - g^{\alpha\beta} \mathcal{L} \right]. \end{aligned} \quad (19)$$

Equation (11) implies that the contribution from the gauge transformations is a total divergence,

$$\begin{aligned} \partial_\gamma \left( \frac{\partial\mathcal{L}}{\partial(\partial_\alpha A^a{}_\gamma)} B^{a\beta} \right) &= \\ \frac{\partial\mathcal{L}}{\partial(\partial_\alpha A^a{}_\gamma)} [\partial_\gamma B^{a\beta} + C^a{}_{bc} B^{c\beta} A^b{}_\gamma] &. \end{aligned} \quad (20)$$

As the variations  $\delta x_\beta$  in (19) are independent, the energy-momentum tensor

$$\begin{aligned} \theta^{\alpha\beta} &= \frac{\partial\mathcal{L}}{\partial(\partial_\alpha A^a{}_\gamma)} [\partial^\beta A^a{}_\gamma - (\partial_\gamma B^{a\beta} + C^a{}_{bc} B^{c\beta} A^b{}_\gamma)] \\ &- g^{\alpha\beta} \mathcal{L} \end{aligned} \quad (21)$$

gives the conserved currents

$$\partial_\alpha \theta^{\alpha\beta} = 0. \quad (22)$$

We demand that  $\theta^{\alpha\beta}$  is symmetric. The Lagrangian term  $g^{\alpha\beta} \mathcal{L}$  is manifestly symmetric and we have to deal with the other contributions. We note that

$$\frac{\partial\mathcal{L}}{\partial(\partial_\alpha A^a{}_\gamma)} = -F^{a\alpha\gamma}$$

with  $F^{a\alpha\gamma}$  given by equation (2). Therefore, the choice

$$B^{a\beta}(x) = A^{a\beta}(x) \quad (23)$$

leads to

$$\theta^{\alpha\beta} = F^{a\alpha\gamma} F^{a\gamma\beta} - g^{\alpha\beta} \mathcal{L} \quad (24)$$

where we used the anti-symmetry of the structure constant under interchange of  $b$  and  $c$ . The tensor (24) is symmetric because of

$$F^{a\alpha\gamma} F^{a\gamma\beta} = F^{a\beta\gamma} F^{a\gamma\alpha}.$$

In conclusion, I have derived the transformation behavior (6) by demanding that the energy-momentum tensor from Noether's theorem comes out symmetric. To the many arguments why gauge invariance is needed, this adds another one: It is needed to make the energy-momentum distribution under local translational variations symmetric.

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